Quantum Field Theory

Set 7: solutions

Exercise 1

The transformation properties of a Weyl fermion under Charge-conjugation are:

$$C^\dagger\,\chi_L\,C = \eta_L\epsilon\chi_R^*,$$
$$C^\dagger\,\chi_R\,C = \eta_R\epsilon\chi_L^*.$$

Let's apply them to the Lagrangian of a Dirac fermion:

$$\begin{aligned}C^\dagger\mathcal{L}\,C &= iC^\dagger\,\chi_L^\dagger\,C\,\bar\sigma^\mu\partial_\mu\,C^\dagger\,\chi_L\,C + iC^\dagger\,\chi_R^\dagger\,C\sigma^\mu\partial_\mu C^\dagger\,\chi_R\,C - m(C^\dagger\,\chi_R^\dagger\,C\,C^\dagger\,\chi_L\,C + h.c.)\\ &= i(\epsilon\chi_R^*)^\dagger\bar\sigma^\mu\partial_\mu\,\epsilon\chi_R^* + i(\epsilon\chi_L^*)^\dagger\sigma^\mu\partial_\mu\,\epsilon\chi_L^* - m(\eta_R^*\eta_L(\epsilon\chi_L^*)^\dagger\epsilon\chi_R^* + h.c)\\ &= i\chi_R^T\epsilon^T\bar\sigma^\mu\partial_\mu\,\epsilon\chi_R^* + i\chi_L^T\epsilon^T\sigma^\mu\partial_\mu\,\epsilon\chi_L^* - m(\eta_R^*\eta_L\chi_L^T\epsilon^T\epsilon\chi_R^* + h.c)\\ &= i\chi_R^T(\sigma^\mu\partial_\mu)^T\,\chi_R^* + i\chi_L^T(\bar\sigma^\mu\partial_\mu)^T\,\chi_L^* - m(\eta_R^*\eta_L\chi_L^T\chi_R^* + h.c).\end{aligned}$$

Where we have used $\epsilon^T(\bar\sigma^\mu)\epsilon = (\mathbb{1}_2, -\epsilon(\bar\sigma^i)\epsilon) = (\sigma^\mu)^T$ and $\epsilon^T\epsilon = \mathbb{1}_2$. At this point we can integrate the Lagrangian by parts (recall that it is the action that must be invariant under a symmetry):

$$C^\dagger\mathcal{L}\,C = -i\partial_\mu\chi_R^T(\sigma^\mu)^T\,\chi_R^* - i\partial_\mu\chi_L^T(\bar\sigma^\mu)^T\,\chi_L^* - m(\eta_R^*\eta_L\chi_L^T\chi_R^* + h.c).$$

In order to simplify we write the indices explicitly:

$$\begin{aligned}C^\dagger\mathcal{L}\,C &= -i\partial_\mu\chi_{R\alpha}(\sigma^\mu)^T_{\alpha\beta}\,\chi^*_{R\beta} - i\partial_\mu\chi_{L\alpha}(\bar\sigma^\mu)^T_{\alpha\beta}\,\chi^*_{L\beta} - m(\eta_R^*\eta_L\chi_{L\alpha}\chi^*_{R\alpha} + h.c)\\ &= -i\partial_\mu\chi_{R\alpha}(\sigma^\mu)_{\beta\alpha}\,\chi^*_{R\beta} - i\partial_\mu\chi_{L\alpha}(\bar\sigma^\mu)_{\beta\alpha}\,\chi^*_{L\beta} - m(\eta_R^*\eta_L\chi_{L\alpha}\chi^*_{R\alpha} + h.c)\\ &= i\chi^*_{R\beta}(\sigma^\mu)_{\beta\alpha}\,\partial_\mu\chi_{R\alpha} + i\chi^*_{L\beta}(\bar\sigma^\mu)_{\beta\alpha}\,\partial_\mu\chi_{L\alpha} + m(\eta_R^*\eta_L\chi^*_{R\alpha}\chi_{L\alpha} + h.c),\end{aligned}$$

where in the last step we have switched the order of the fermions and used the fact that two fermions anti-commute. Finally (up to total derivatives):

$$C^\dagger\mathcal{L}\,C = i\chi_L^\dagger\,\bar\sigma^\mu\partial_\mu\,\chi_L + i\chi_R^\dagger\sigma^\mu\partial_\mu\chi_R + m(\eta_R^*\eta_L\chi_R^\dagger\chi_L + \eta_R\eta_L^*\chi_L^\dagger\chi_R).$$

We see that the only way to achieve the invariance of the Dirac action is to impose $\eta_R^*\eta_L = -1$.
Note that this condition can also be easily obtained by noting that, applying twice the charge conjugation operator on a Weyl spinor, one should get back the spinor itself: $C^\dagger C^\dagger\chi_L CC = C^\dagger\eta_L\epsilon\chi_R^*C = \eta_L\eta_R^*\epsilon^2\chi_L = \chi_L$, which implies $\eta_R^*\eta_L = -1$ since $\epsilon^2 = -1$. Note also that, in order to satisfy the physical requirement $C^2 = 1$, it must be $C = C^\dagger$ (since C is unitary), as it is for parity.
On a Dirac spinor, the action of charge conjugation is

$$C^\dagger\begin{pmatrix}\chi_L\\ \chi_R\end{pmatrix}C = \underbrace{\begin{pmatrix}-\eta_L & 0\\ 0 & \eta_R\end{pmatrix}}_{\eta_C} i\underbrace{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^2\\ -\sigma^2 & 0\end{pmatrix}}_{\gamma^0\gamma^2}\underbrace{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\chi_L^*\\ \chi_R^*\end{pmatrix}}_{\bar\psi^T},$$

where we have used $\sigma^2 = -i\epsilon$. This proves that $U_C = i\gamma^0\gamma^2$. Note that the choice $\eta_L = -1$, $\eta_R = 1$, compatible with the constraint $\eta_R^*\eta_L = -1$, the matrix η_C can be eliminated from the formalism since it becomes the identity.

Exercise 2

Let's repeat the steps of Exercise 2 of Set 19, for charge conjugation. We recall the definition of charge conjugation on the creation and annihilation operators, deduced in Exercise 1,

$$Ca^\dagger(\vec k)C = \eta_C^*b^\dagger(\vec k), \qquad Cb^\dagger(\vec k)C = \eta_C a^\dagger(\vec k),$$

and write explicitly

$$C|\Phi_l\rangle = \int d\Omega_{\vec{p}}\; f_l(\vec{p},-\vec{p})\, C a^\dagger(\vec{p}) C\, C b^\dagger(-\vec{p}) C|0\rangle$$
$$= \eta_C \eta_C^* \int d\Omega_{\vec{p}}\; f_l(\vec{p},-\vec{p})\, b^\dagger(\vec{p})\, a^\dagger(-\vec{p})|0\rangle$$
$$= \int d\Omega_{\vec{p}}\; f_l(-\vec{p},\vec{p})\, a^\dagger(\vec{p})\, b^\dagger(-\vec{p})\,|0\rangle = (-1)^l|\Phi_l\rangle,$$

where we have repositioned $a^\dagger$ and $b^\dagger$ in the initial order using the commutation relation $[a^\dagger,\, b^\dagger] = 0$.
Note that the combined action of the two transformations leaves a state of scalar particle-antiparticle invariant:

$$CP|\Phi_l\rangle = (-1)^{l+l}|\Phi_l\rangle = |\Phi_l\rangle.$$

Let's now move to fermions, recalling that

$$C b^\dagger(r,\vec{k}) C = -\eta_C^* \tilde{d}^\dagger(r,\vec{k}) \quad \Longrightarrow \quad C\tilde{d}^\dagger(r,\vec{k})C = -\eta_C b^\dagger(r,\vec{k}).$$

One has

$$C|\Psi_{l,S}\rangle = \sum_{r,t}\int d\Omega_{\vec{p}}\; f_l(\vec{p},-\vec{p})\,\chi_S(r,t)\, C b^\dagger(t,\vec{p}) C C\tilde{d}^\dagger(r,-\vec{p})C|0\rangle$$
$$= \sum_{r,t}\int d\Omega_{\vec{p}}\; f_l(\vec{p},-\vec{p})\,\chi_S(r,t)\,\tilde{d}^\dagger(t,\vec{p})\, b^\dagger(r,-\vec{p})|0\rangle$$
$$= -\sum_{r,t}\int d\Omega_{\vec{p}}\; f_l(\vec{p},-\vec{p})\,\chi_S(r,t)\, b^\dagger(r,-\vec{p})\,\tilde{d}^\dagger(t,\vec{p})|0\rangle = (-1)^{l+S}|\Psi_{l,S}\rangle,$$

where the minus sign in the third line comes from changing the order of the operators, since they anticommute $\{\tilde{d}^\dagger,\, b^\dagger\} = 0$.
In the end, the combined action of the two transformations on a state made of fermionic particle-antiparticle is given by:

$$CP|\Psi_{l,S}\rangle = (-1)^{l+S}(-1)^{l+1}|\Psi_{l,S}\rangle = (-1)^{S+1}|\Psi_{l,S}\rangle.$$

Exercise 3

Given the transformation properties of a Dirac fermion ψ under charge conjugation ($C = C^\dagger$), namely

$$C\psi(t,\vec{x})C = -i\eta_C\gamma^2\psi^*(t,\vec{x}),$$

we want to compute the transformation properties of all the bilinears of the form $\bar{\psi}\Gamma\psi$, where Γ is some 4×4 matrix. In order to do this, it is sufficient to compute the transformation properties for

$$\Gamma = \{1_4, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \gamma^{\mu\nu}\},$$

since we have proved that any 4×4 matrix can be decomposed into a linear combination of these quantities.
Before proceeding further, it is useful to work out a close form for $(\gamma^\mu)^T$. The expression for the gamma matrices is

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

Making use of the Clifford algebra of the gamma matrices, $\{\gamma^\mu,\, \gamma^\nu\} = 2\eta^{\mu\nu}$, one can guess a formula for $(\gamma^\mu)^T$: γ^0, γ^2 are symmetric while γ^1, γ^3 are antisymmetric. Hence

$$\gamma^0\gamma^2\gamma^\mu\gamma^2\gamma^0 = \begin{cases} \gamma^0\gamma^2\gamma^0\gamma^2\gamma^0 = -(\gamma^2)^2\gamma^0 = \gamma^0 & \mu = 0, \\ \gamma^0(\gamma^2)^3\gamma^0 = -\gamma^0\gamma^2\gamma^0 = \gamma^2 & \mu = 2, \\ \gamma^0\gamma^2\gamma^i\gamma^2\gamma^0 = \gamma^0\gamma^2\gamma^2\gamma^0\gamma^i = -\gamma^i & \mu = i = 1,3, \end{cases}$$

which is $= (\gamma^\mu)^T$.
Let's manipulate a bit the transformation under C:

$$C\psi C = -i\eta_C\gamma^2\psi^* = -i\eta_C\gamma^2(\psi^\dagger)^T = -i\eta_C(\psi^\dagger\gamma^2)^T = -i\eta_C(\bar{\psi}\gamma^0\gamma^2)^T.$$

Similarly:

$$C\bar{\psi}C = C\psi^\dagger C\gamma^0 = -i\eta_C^*\psi^T\gamma^2\gamma^0 = -i\eta_C^*(\gamma^0\gamma^2\psi)^T.$$

Now we are able to compute the transformation properties of the bilinears:

$$C\bar{\psi}\Gamma\psi C = -|\eta_C|^2(\gamma^0\gamma^2\psi)^T\Gamma(\bar{\psi}\gamma^0\gamma^2)^T.$$

A bilinear $\bar{\psi}\Gamma\psi$ is a number, since all the spinorial indices are contracted. This means that it is equal to its transpose, however we must pay attention to ordering, since fermions anticommute:

$$\begin{aligned} C\bar{\psi}\Gamma\psi C &= -(\gamma^0\gamma^2)_{im}\psi_m\Gamma_{ij}\bar{\psi}_n(\gamma^0\gamma^2)_{nj} = (\gamma^0\gamma^2)_{im}\bar{\psi}_n\psi_m\Gamma_{ij}(\gamma^0\gamma^2)_{nj} \\ &= \bar{\psi}_n(\gamma^0\gamma^2)_{nj}\Gamma_{ij}(\gamma^0\gamma^2)_{im}\psi_m = \bar{\psi}(\gamma^0\gamma^2)\Gamma^T(\gamma^0\gamma^2)\psi = -\bar{\psi}\gamma^2\gamma^0\Gamma^T\gamma^0\gamma^2\psi \end{aligned}$$

Everything is now reduced to understanding what $-\gamma^2\gamma^0\Gamma^T\gamma^0\gamma^2$ is. Let's see it case by case:

- Let's start from the simplest case: $\Gamma = 1_4$. Then:

  $$-\gamma^2\gamma^0\gamma^0\gamma^2 = 1.$$

  This means that:

  $$C\bar{\psi}\psi C = \bar{\psi}\psi.$$

- Let us consider now $\Gamma = \gamma^5$. Hence

  $$-\gamma^2\gamma^0\gamma^5\gamma^0\gamma^2 = \gamma^5.$$

  This means that:

  $$C\bar{\psi}\gamma^5\psi C = \bar{\psi}\gamma^5\psi.$$

- The next one is $\Gamma = \gamma^\mu$:

  $$-\gamma^2\gamma^0(\gamma^\mu)^T\gamma^0\gamma^2 = -\gamma^2\gamma^0\gamma^0\gamma^2\gamma^\mu\gamma^2\gamma^0\gamma^0\gamma^2 = -\gamma^\mu.$$

  This means that

  $$C\bar{\psi}\gamma^\mu\psi C = -\bar{\psi}\gamma^\mu\psi$$

- The following term is $\Gamma = \gamma^\mu\gamma^5$:

  $$\begin{aligned} -\gamma^2\gamma^0(\gamma^\mu\gamma^5)^T\gamma^0\gamma^2 &= -\gamma^2\gamma^0\gamma^5(\gamma^\mu)^T\gamma^0\gamma^2 \\ &= -\gamma^5\gamma^2\gamma^0(\gamma^\mu)^T\gamma^0\gamma^2 = -\gamma^5\gamma^\mu = \gamma^\mu\gamma^5. \end{aligned}$$

  This means that

  $$C\bar{\psi}\gamma^\mu\gamma^5\psi C = \bar{\psi}\gamma^\mu\gamma^5\psi.$$

- The last term is $\Gamma = \gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$:

  $$-\gamma^2\gamma^0(\gamma^{\mu\nu})^T\gamma^0\gamma^2 = -\frac{1}{2}\gamma^2\gamma^0[\gamma^{\nu\,T}, \gamma^{\mu\,T}]\gamma^0\gamma^2.$$

  Notice that

  $$[\gamma^{\nu\,T}, \gamma^{\mu\,T}] = \gamma^0\gamma^2\gamma^\nu\gamma^2\gamma^0\gamma^0\gamma^2\gamma^\mu\gamma^2\gamma^0 - (\mu \leftrightarrow \nu) = -\gamma^0\gamma^2[\gamma^\nu, \gamma^\mu]\gamma^2\gamma^0 = \gamma^0\gamma^2[\gamma^\mu, \gamma^\nu]\gamma^2\gamma^0,$$

  so that

  $$-\gamma^2\gamma^0(\gamma^{\mu\nu})^T\gamma^0\gamma^2 = -\gamma^{\mu\nu}.$$

  This means that

  $$C\bar{\psi}\gamma^{\mu\nu}\psi C = -\bar{\psi}\gamma^{\mu\nu}\psi.$$

  Due to its transformation under parity, this object is called a tensor.

Table 1: Summary of bilinear transformations.

	∂_μ	$\bar\psi\psi$	$\bar\psi\gamma^5\psi$	$\bar\psi\gamma^\mu\psi$	$\bar\psi\gamma^\mu\gamma^5\psi$	$\bar\psi\gamma^{\mu\nu}\psi$
P	$\mathscr{P}^\mu{}_\rho$	1	-1	$\mathscr{P}^\mu{}_\rho$	$-\mathscr{P}^\mu{}_\rho$	$\mathscr{P}^\mu{}_\rho\mathscr{P}^\nu{}_\sigma$
C	1	1	1	-1	1	-1
T	$-\mathscr{P}^\mu{}_\rho$	1	-1	$\mathscr{P}^\mu{}_\rho$	$\mathscr{P}^\mu{}_\rho$	$-\mathscr{P}^\mu{}_\rho\mathscr{P}^\nu{}_\sigma$

Let us summarize the transformations of bilinears under P (see exercise 3 of set 20) and C. We will include also the transformation properties of ∂_μ: this is because we will make use of the CPT invariance to infer the transformation of the above bilinears under time reversal T. However the CPT theorem applies only to Lorentz invariant operator, therefore, when needed, we must contract with the derivative. All transformation properties are summarized in the table. One can verify that all the Lorentz invariant operators that can be constructed satisfy $CPT = 1$.

Note that the compact notation regarding the transformation properties of the derivative in the table actually means that $P\partial_\mu SP = \mathscr{P}^\mu{}_\rho\partial^\rho S$ (and similarly for C and T), where S is a scalar made up with fields: indeed P, C, and T act non trivially only on fields. We have $\mathscr{P} = \mathrm{diag}(1, -1, -1, -1)$.

Exercise 4: local interactions and superposition principle

Let us consider an orthonormalized one particle state

$$|\Psi\rangle = \int d\Omega_k \hat f(k) a_k^\dagger |0\rangle.$$

We want to compute the expectation value $\langle\Psi| : \varphi(x)^2 : |\Psi\rangle$, of the normal ordered field:

$$: \varphi(x)^2 :\equiv \varphi_+^2 + \varphi_-^2 + 2\varphi_+\varphi_-,$$

where

$$\varphi_+(x) = \int d\Omega_k\, e^{ikx} a_k^\dagger, \qquad \varphi_-(x) = \int d\Omega_k\, e^{-ikx} a_k = (\varphi_+(x))^\dagger .$$

First let us notice that the first term φ_+^2 contains two creation creation operators, hence the state $\varphi_+^2|\Psi\rangle$ contain three particles and cannot have any overlap with a one-particle state:

$$\langle\Psi|\varphi_+^2|\Psi\rangle = \langle\Psi|\varphi_-^2|\Psi\rangle^* = 0.$$

Then we just need to compute:

$$2\langle\Psi|\varphi_+\varphi_-|\Psi\rangle = 2\|\varphi_-|\Psi\rangle\|^2.$$

To evaluate the latter, recall

$$\int d\Omega_k\, [a_k, a_p^\dagger] = \int d^3k\, \delta^3(\vec k - \vec p),$$

whence we obtain

$$\begin{aligned}\varphi_-|\Psi\rangle &= \int d\Omega_k\, e^{-ikx}\, a_k \int d\Omega_p\, \hat f(p)\, a_p^\dagger|0\rangle = \iint d\Omega_k d\Omega_p\, e^{-ikx}\hat f(p)\, [a_k, a_p^\dagger]|0\rangle \\ &= \int d\Omega_k\, e^{-ikx}\hat f(k)|0\rangle = f(x)|0\rangle,\end{aligned}$$

where we defined the direct space wave-function as:

$$f(x) \equiv \int d\Omega_k\, e^{-ikx}\hat f(k).$$

Then we find immediately:

$$\langle\Psi| : \varphi(x)^2 : |\Psi\rangle = 2\|\varphi_-|\Psi\rangle\|^2 = 2|f(x)|^2.$$

Consider now an orthonormalized two particle state

$$|\Psi\rangle = \int d\Omega_1 d\Omega_2 \hat{f}_1(k_1)\hat{f}_2(k_2) a^\dagger_{k_1} a^\dagger_{k_2}|0\rangle.$$

Let us suppose that the wave-packets $\hat{f}_1(k_1)$ and $\hat{f}_2(k_2)$ are spacially separated. This is achieved taking, for instance,

$$f_1(k) = f_2(k) e^{i\vec{R}\cdot\vec{k}}, \tag{1}$$

so that one finds the direct space packet:

$$\int d\Omega\, e^{-ikx}\hat{f}_1(k) = \int d\Omega\, e^{-ikx}\hat{f}_2(k) e^{i\vec{R}\cdot\vec{k}} = f_2(t, \vec{x}+\vec{R}). \tag{2}$$

As long as $f_1(x)$ has compact support, this is spatially separated for $|\vec{R}| \to \infty$.
Consider first the norm of the vector:

$$\begin{aligned}
\langle\Psi|\Psi\rangle &= \int d\Omega_3 d\Omega_4 d\Omega_1 d\Omega_2\, \hat{f}^*_1(k_3)\hat{f}^*_2(k_4)\hat{f}_1(k_1)\hat{f}_2(k_2)\, \langle 0|a_{k_3}a_{k_4}a^\dagger_{k_1}a^\dagger_{k_2}|0\rangle \\
&= \int d\Omega_3 d\Omega_4 d\Omega_1 d\Omega_2\, \hat{f}^*_1(k_3)\hat{f}^*_2(k_4)\hat{f}_1(k_1)\hat{f}_2(k_2) \left(\langle 0|a_{k_3}[a_{k_4}, a^\dagger_{k_1}]a^\dagger_{k_2}|0\rangle + \langle 0|a_{k_3}a^\dagger_{k_1}[a_{k_4}, a^\dagger_{k_2}]|0\rangle\right) \\
&= \int d\Omega_3 d\Omega_1 d\Omega_2\, \hat{f}^*_1(k_3)\hat{f}^*_2(k_1)\hat{f}_1(k_1)\hat{f}_2(k_2)\langle 0|[a_{k_3}, a^\dagger_{k_2}]|0\rangle + \int d\Omega_3 d\Omega_1 d\Omega_2\, \hat{f}^*_1(k_3)\hat{f}^*_2(k_2)\hat{f}_1(k_1)\hat{f}_2(k_2)\, \langle 0|[a_{k_3}, a^\dagger_{k_1}]|0\rangle \\
&= \left|\int d\Omega\, \hat{f}^*_1(k)\hat{f}_2(k)\right|^2 + \left(\int d\Omega_1 \left|\hat{f}_1(k_1)\right|^2\right)\left(\int d\Omega_2\, \left|\hat{f}_2(k_2)\right|^2\right).
\end{aligned}$$

Notice that the first term comes indeed because we have two equal particles. Now note that by Riemann-Lebesgue lemma

$$\int d\Omega\, \hat{f}^*_1(k)\hat{f}_2(k) = \int d\Omega\, e^{-i\vec{R}\cdot\vec{k}}\, \left|\hat{f}_2(k)\right|^2 \overset{|\vec{R}|\to\infty}{\longrightarrow} 0.$$

Then in this limit

$$\langle\Psi|\Psi\rangle = \left(\int d\Omega_1 \left|\hat{f}_1(k_1)\right|^2\right)\left(\int d\Omega_2\, \left|\hat{f}_2(k_2)\right|^2\right).$$

For a normalized wave-packet we can always require:

$$\int d\Omega_1 \left|\hat{f}_1(k_1)\right|^2 = \int d\Omega_2\, \left|\hat{f}_2(k_2)\right|^2 = 1. \tag{3}$$

Now consider the expectation value on this two particle state of $:\varphi(x)^2:$. By the same considerations done for the one-particle state, we get:

$$\langle\Psi| :\varphi(x)^2: |\Psi\rangle = 2\|\varphi_-|\Psi\rangle\|^2.$$

The computation proceeds as before

$$\begin{aligned}
\varphi_-|\Psi\rangle &= \int d\Omega\, d\Omega_1 d\Omega_2\, e^{-ikx}\hat{f}_1(k_1)\hat{f}_2(k_2)\, a_k a^\dagger_{k_1}a^\dagger_{k_2}|0\rangle \\
&= \int d\Omega\, d\Omega_1 d\Omega_2\, e^{-ikx}\hat{f}_1(k_1)\hat{f}_2(k_2)\, \left([a_k, a^\dagger_{k_1}]a^\dagger_{k_2}|0\rangle + a^\dagger_{k_1}[a_k, a^\dagger_{k_2}]|0\rangle\right) \\
&= f_1(x)|\Psi_2\rangle + f_2(x)|\Psi_1\rangle,
\end{aligned}$$

where we defined

$$f_i(x) = \int d\Omega \hat{f}_i(k)\, e^{-ikx}, \qquad |\Psi_i\rangle = \int d\Omega\, \hat{f}_i(k)\, a^\dagger_{k_i}|0\rangle.$$

Then:

$$\|\varphi_-|\Psi\rangle\|^2 = |f_1(x)|^2\||\Psi_1\rangle\|^2 + |f_2(x)|^2\||\Psi_1\rangle\|^2 + \left(f^*_1(x)f_2(x)\langle\Psi_2|\Psi_1\rangle + c.c.\right).$$

The first two terms, using the normalization (3), are just the sum of the expectation values for two single particle-states. The last term instead violates the superimposition principle. However for spatially separated wave packets we have $f_1^*(x)f_2(x) \simeq 0$ and also

$$\langle\Psi_2|\Psi_1\rangle = \int d\Omega\, \hat{f}_1^*(k)\hat{f}_2(k) = \int d\Omega\, e^{-i\vec{R}\cdot\vec{k}} \left|\hat{f}_2(k)\right|^2 \overset{|\vec{R}|\to\infty}{\longrightarrow} 0.$$

Hence in this limit the superimposition principle holds:

$$\langle\Psi| : \varphi(x)^2 : |\Psi\rangle = 2\left(|f_1(x)|^2 + |f_2(x)|^2\right).$$

Finally consider

$$: \varphi(x)^4 :\equiv \varphi_+^4 + 4\varphi_+^3\varphi_- + 6\varphi_+^2\varphi_-^2 + 4\varphi_+\varphi_-^3 + \varphi_-^4.$$

This is an interesting interaction term to study. The computation is analogous to the previous ones:

$$\langle\Psi| : \phi(x)^4 : |\Psi\rangle = 6\|\phi_-^2|\psi\rangle\|^2,$$

$$\begin{aligned}\phi_-^2|\psi\rangle &= \int d\Omega_3 d\Omega_4 d\Omega_1 d\Omega_2\, e^{-ik_3x}e^{-ik_4x}\, a_{k_3}a_{k_4}a_{k_1}^\dagger a_{k_2}^\dagger|0\rangle \\ &= \ldots = 2f_1(x)f_2(x).\end{aligned}$$

Then

$$\langle\Psi| : \phi(x)^4 : |\Psi\rangle = 6\|\phi_-^2|\psi\rangle\|^2 = 24|f_1(x)|^2|f_2(x)|^2.$$

If two wave packets are spatially separeted this vanishes.

Let us now consider the specific case of two **time-evolved** wave-packets with shapes

$$\hat{f}_1(\vec{k}) \propto \exp\left(-\frac{(\vec{k}-\vec{k}_0)^2}{2\Delta^2} + i\vec{k}\cdot\vec{a}\right) \qquad \hat{f}_2(\vec{k}) \propto \exp\left(-\frac{(\vec{k}+\vec{k}_0)^2}{2\Delta^2} - i\vec{k}\cdot\vec{a}\right) \tag{4}$$

Namely, we will consider the state

$$|\Psi(t)\rangle = \int d\Omega_{\vec{k}_1} d\Omega_{\vec{k}_2} e^{-i\omega(k_1)t}\hat{f}_1(\vec{k}_1)e^{-i\omega(k_2)t}\hat{f}_2(\vec{k}_2)a_{k_1}^\dagger a_{k_2}^\dagger|0\rangle \tag{5}$$

where we have applied time evolution e^{-iHt} and the energy is given by $\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$ for a particle of mass m. The computations are analogous to the previous ones. For the norm, we obtain the same result as the previous one by time translation invariance,

$$\langle\Psi(t)|\Psi(t)\rangle = \langle\Psi(0)|\Psi(0)\rangle = \left|\int d\Omega \hat{f}_1^*(\vec{k})\hat{f}_2(\vec{k})\right| + \left(\int d\Omega_1|\hat{f}_1(\vec{k}_1)|^2\right)\left(\int d\Omega_2|\hat{f}_2(\vec{k}_2)|^2\right) \tag{6}$$

For the specific shapes (4), the first term still vanishes when $|\vec{a}| \to \infty$ which corresponds to very separated states at a fixed value of time. However, in general, it doesn't. Now consider the more interesting correlator

$$\langle\Psi(t)| : \varphi(0,\vec{x})^2 : |\Psi(t)\rangle \tag{7}$$

where for simplicity we've set the time component of ϕ to 0. Proceeding as before, one needs to compute

$$\langle\Psi(t)| : \varphi(0,\vec{x})^2 : |\Psi(t)\rangle = ||\varphi_-(0,\vec{x})|\Psi(t)\rangle||^2. \tag{8}$$

where one obtains

$$\begin{aligned}\varphi_-(0,\vec{x})|\Psi(t)\rangle = &\underbrace{\int d\Omega_{\vec{k}} e^{-i\omega(\vec{k})t + i\vec{k}\cdot\vec{x}}\hat{f}_1(\vec{k})}_{\equiv f_1(t,\vec{x})}\underbrace{\int d\Omega_{\vec{k}_2} e^{-i\omega(\vec{k}_2)t}\hat{f}_2(\vec{k}_2)a_{\vec{k}_2}^\dagger|0\rangle}_{\equiv|\Psi_2(t)\rangle} \\ &+ \underbrace{\int d\Omega_{\vec{k}} e^{-i\omega(\vec{k})t + i\vec{k}\cdot\vec{x}}\hat{f}_2(\vec{k})}_{\equiv f_2(t,\vec{x})}\underbrace{\int d\Omega_{\vec{k}_1} e^{-i\omega(\vec{k}_1)t}\hat{f}_1(\vec{k}_1)a_{\vec{k}_1}^\dagger|0\rangle}_{\equiv|\Psi_1(t)\rangle}\end{aligned} \tag{9}$$

Thus, we now have

$$\langle\Psi(t)|:\varphi(0,\vec{x})^2:|\Psi(t)\rangle = |f_1(t,\vec{x})|^2\langle\Psi_2(t)|\Psi_2(t)\rangle + |f_2(t,\vec{x})|^2\langle\Psi_1(t)|\Psi_1(t)\rangle + (f_1^*(t,\vec{x})f_2(t,\vec{x})\langle\Psi_1(t)|\Psi_2(t)\rangle + \text{c.c.}) \tag{10}$$

Here we notice that the term that violates the superimposition principle is time-dependent. If at $t \to \pm\infty$, the shapes have no overlap in position space, then this term vanishes, as expected physically. Let us examine the specific shapes (4). The goal will be to understand the form of $f_i(t,\vec{x})$ when $\hat{f}_i$ are peaked around $\pm\vec{k}_0$ i.e. when Δ is small compared to the other scales of the problem. Let us start with $f_1(t,\vec{x})$. We thus compute

$$f_1(t,\vec{x}) \propto \int d\Omega_{\vec{k}} \exp\left(-\frac{|\vec{k}-\vec{k}_0|^2}{2\Delta^2} + i\vec{k}\cdot(\vec{a}+\vec{x}) - i\omega(\vec{k})t\right) \tag{11}$$

in the limit of small Δ. Let us change variables from $\vec{k}$ to $\vec{q}$ by $\vec{k} = \vec{k}_0 + \vec{q}\,\Delta$ and expand the exponent up to the first non-trivial term in Δ. We obtain

$$f_1(t,\vec{x}) \propto e^{i\vec{k}_0\cdot(\vec{x}+\vec{a})-i\omega(\vec{k}_0)t}\int d\Omega_{\vec{k}_0+\vec{q}\Delta}\exp\left(-\frac{q^2}{2} - it\Delta\left.\frac{\partial\omega(\vec{k})}{\partial\vec{k}}\right|_{\vec{k}=\vec{k}_0}\cdot\vec{q} + i\Delta\vec{q}\cdot(\vec{x}+\vec{a}) + \dots\right) \tag{12}$$

Now complete the square in the exponent, namely

$$-\frac{q^2}{2} - it\Delta\left.\frac{\partial\omega(\vec{k})}{\partial\vec{k}}\right|_{\vec{k}=\vec{k}_0}\cdot\vec{q} + i\Delta\vec{q}\cdot(\vec{x}+\vec{a}) = -\frac{1}{2}\left(\vec{q} - i\Delta\left(\vec{x}+\vec{a}-\frac{\partial\omega}{\partial\vec{k}}t\right)\right)^2 - \frac{1}{2}\Delta^2\left(\vec{x}+\vec{a}-\frac{\partial\omega}{\partial\vec{k}}t\right)^2 \tag{13}$$

Ignoring the measure factor[1], the integral over $\vec{q}$ is Gaussian, and thus produces a number independent from t and $\vec{x}$. We're thus left with

$$f_1(t,\vec{x}) \propto e^{i\vec{k}_0\cdot(\vec{x}+\vec{a})-i\omega(\vec{k}_0)t}\exp\left(-\frac{1}{2}\Delta^2\left(\vec{x}+\vec{a}-\frac{\partial\omega}{\partial\vec{k}}t\right)^2\right) \tag{14}$$

In our relativistic case,

$$\left.\frac{\partial\omega}{\partial\vec{k}}\right|_{\vec{k}_0} = \frac{\vec{k}_0}{\sqrt{\vec{k}_0^2+m^2}} = \vec{\beta}_0 \tag{15}$$

is the expected velocity β_0 of a particle of mass m moving with momentum $\vec{k}_0$. Note that to compute $f_2(t,\vec{x})$, one simply needs to change the sign $\vec{k}_0 \to -\vec{k}_0$ and $\vec{a} \to -\vec{a}$. We thus obtained,

$$f_1(t,\vec{x}) \propto e^{i\vec{k}_0\cdot(\vec{x}+\vec{a})-i\omega(\vec{k}_0)t}\exp\left(-\frac{1}{2}\Delta^2\left(\vec{x}+\vec{a}-\vec{\beta}_0 t\right)^2\right) \tag{16}$$

$$f_2(t,\vec{x}) \propto e^{-i\vec{k}_0\cdot(\vec{x}-\vec{a})-i\omega(\vec{k}_0)t}\exp\left(-\frac{1}{2}\Delta^2\left(\vec{x}-\vec{a}+\vec{\beta}_0 t\right)^2\right) \tag{17}$$

In position space, $f_1(\vec{x},t)$ is thus peaked at $\vec{x} = -\vec{a} + \vec{\beta}_0 t$ while $f_2(t,\vec{x})$ is peaked at $\vec{x} = \vec{a} - \vec{\beta}_0 t$. Both have a width $\delta x \sim 1/\Delta$. This allows to draw the shapes of f_1 and f_2 as shown in Figure 1 below. It is clear for this example that as $t \to \pm\infty$, f_1 and f_2 have no overlap, and thus the superimposition principle holds, even for finite $\vec{a}$.

[1] To be precise, one can multiply $\hat{f}_1$ by a pre-factor $2\omega(\vec{k})$ to cancel the measure term in $d\Omega_{\vec{k}}$

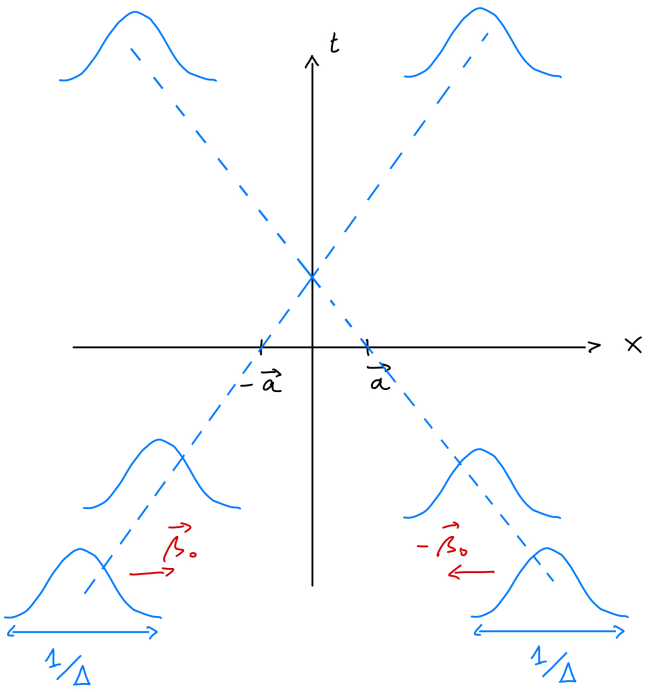

Figure 1: Time evolution of the gaussian shapes

Digression on normal ordering

Products of fields at the same point are mathematically ill defined in general. Indeed fields are distribution valued operators and the product of distributions is not well defined. This does not mean that we cannot make sense at all of them in a quantum field theory, but simply that a prescription to make sense of objects like $\varphi(x)^2$ is needed. In other words, we have to *define* their meaning in a consistent way.
A simple example of the need of such a prescription is found computing the Hamiltonian of a free scalar field theory. A straightforward computation (see chap. 4 of the lecture notes) leads to the expression:

$$H = \frac{1}{2}\int d^3x \left[\dot{\varphi}(x)^2 + \left(\vec{\nabla}\varphi(x)\right)^2 + m^2\varphi(x)^2\right] = \frac{1}{2}\int d\Omega_k\, \omega_k \left(a_k^\dagger a_k + a_k a_k^\dagger\right).$$

Now using the commutation rules $[a_k, a_p^\dagger] = (2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{p})$, we get:

$$H = \int d\Omega_k\, \omega_k\, a_k^\dagger a_k + (2\pi)^3\delta^3(0)\int \frac{d^3k}{(2\pi)^3}\,\frac{\omega_k}{2}.$$

The second term is divergent, even when performing the substitution $(2\pi)^3\delta^3(0) \to V$. This is called the *zero-point energy* or *vacuum energy*. We can *regulate* the divergence putting a *cutoff* Λ over large momenta, so that we integrate only over $|\vec{p}| < \Lambda$. This is motivated by the fact that we cannot observe particles at arbitrary high energy in experiments, hence we expect our theory to be valid only at energies below a certain cutoff. In this way we get the vacuum energy density:

$$\rho_{vac} \sim \int^\Lambda p^3 dp \sim \Lambda^4.$$

In this case the divergence however is relatively harmless. Since what we measure are energy differences[2], we can simply discard zero-point energy and declare that our Hamiltonian is

$$H = \int d\Omega_k\, \omega_k\, a_k^\dagger a_k. \tag{18}$$

We can formalize our discussion, saying that the Hamiltonian involve products of fields at coincident points and this leads to a diverging result; however we can still make sense of it, for instance introducing a cutoff. The natural

[2]This is true only as long as we do not consider gravity; in general relativity the zero-point energy might act as a *cosmological constant* term.

prescription that gives (18) corresponds to defining products of operator through the *normal ordering*: given an operator $\mathcal{O}$, we define its normal ordered form $:\mathcal{O}:$ as the operator obtained writing by hand all creation operators to the left of all destruction operators. Thus for instance $:a_p a_p^\dagger := a_p^\dagger a_p$ and we immediately get

$$:H:=\int d\Omega_k\,\omega_k\,a_k^\dagger a_k.$$